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On the structure of Arnold tongues

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joint work with
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October 9, 2025

Content

Synchronization

- synchronization of heartbeat, respiration and motion in mammals, Josephson junctions, neural activity, cellular oscillators, ...
- Deeper understanding of the mechanism of frequency locking in dynamical systems can lead to advances in knowledge in many scientific fields.
- Arnold tongue is one of the mathematical concepts that explain and quantify the onset of synchronization.

Coupled oscillators

- Coupling oscillators can lead to various types of synchronization.
- How can we interpret these solutions? How do they characterize the behaviour of the system?

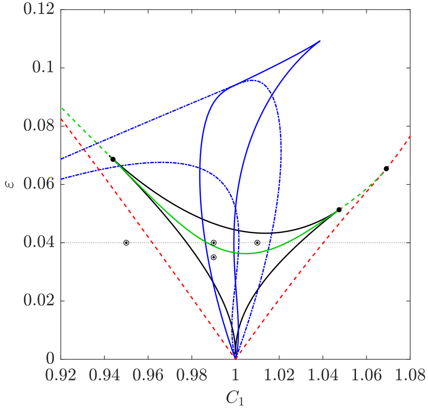


Figure: Three Arnold tongues (two blue, one black). The figure is taken from [4].

Neimark-Sacker bifurcation

- Neimark-Sacker bifurcation adds a new frequency to the solution - if this happens in a neighbourhood of a resonance point, a $(r : q)$-synchronization may occur.
- How can we (algorithmically) continuate borders of regions of synchronization?

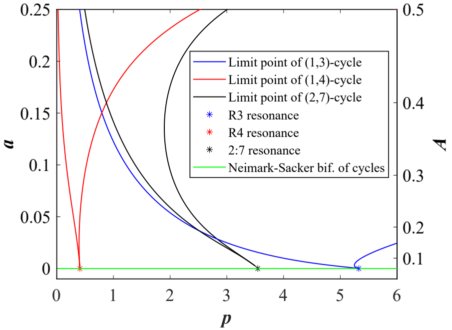


Figure: Three Arnold tongues in the vicinity of Neimark-Sacker bifurcation. The figure is taken from [5].

Two identical coupled oscillators

Let $f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ be a smooth function and suppose that

$$\dot{x}(t) = f(x(t), c), \tag{1}$$

has a p-periodic non-degenerate stable solution $x_0(t)$ for $c = c_0$. We will study the following system:

$$\begin{aligned} \dot{x}(t) - f(x(t), c_1) - K_1(x_t, y_t, \lambda, c_1, c_2) &= 0, \\ \dot{y}(t) - f(y(t), c_2) - K_2(x_t, y_t, \lambda, c_1, c_2) &= 0, \end{aligned} \tag{2}$$

where $x_t \in C([-r, 0], \mathbb{R}^k)$, $r > 0$ and $x_t(\theta) = x(t + \theta)$ is called history. We assume that $K(\cdot, \cdot, 0, \cdot, \cdot) \equiv 0$ and that the coupling is symmetric:

$$K_1(x_t, y_t, \lambda, c_1, c_2) = K_2(y_t, x_t, \lambda, c_2, c_1).$$

Torus of solutions

Obviously the system (2) has a solution

$$(x(t), y(t)) = (x_0(t + \alpha + \beta), x_0(t + \alpha - \beta)) \tag{3}$$

for $\lambda = 0$, $c_1, c_2 = c_0$ where $\alpha, \beta \in \mathbb{R}$ are arbitrary. These solutions form a torus in the Banach space C^1_p of p-periodic differentiable functions $u\colon \mathbb{R} \to \mathbb{R}^{2k}$. The following is the main result of this section which states that the torus is persistent:

Theorem

There is $\varepsilon > 0$ and $M = M(\alpha, \beta, \lambda)\colon \mathbb{R}^2 \times (-\varepsilon, \varepsilon) \to C^1_p$ such that M is periodic in the first two variables and it parametrizes all periodic solutions of (2) in a neighbourhood of the family (3).

Two coupled interneurons

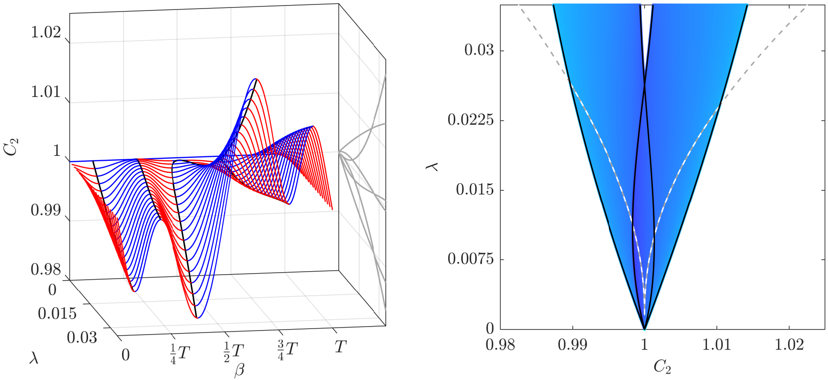


Figure: Projection of folds of manifold of periodic solutions form boundaries of synchronization regions. The figure is taken from [6].

Notes on numerical methods

- For a fixed value of the coupling parameter λ, periodic solutions can be numerically continuated with respect to one parameter and period using standard techniques.
- Parameter β is given implicitly - it has to be estimated.
- We can minimize the L_2 norm of the difference between the solution components of the first and second oscillators, shifted by a phase shift $\hat{\beta}$, with respect to $\hat{\beta}$. The value of $\hat{\beta}$ for which the minimum is reached, is used as the estimate for the true value of β.

Remarks

- The parameter β can be interpreted as a phase shift between the two oscillators for λ small $\rightarrow$ this reveals the structure of solutions inside the tongue.
- Calculation of $\hat{\beta}$ enables to estimate the interval for phase shift for which the stable synchronization is possible.
- Symmetry of the coupling and equity of oscillators was used just for the proof of existence of in-phase and anti-phase oscillations.
- The existence of manifold of periodic solutions can be proven for any finite number of general (possibly different) oscillators with rationally dependent periods.

Germs, equivariance and equivalence of germs

- A germ is an equivalence class of functions $f\colon \mathbb{R}^n \to \mathbb{R}^n$ coinciding on some neighbourhood of 0.
- Let Γ be a group having a linear action on $\mathbb{R}^n$.
- A germ g is Γ-equivariant if $g(\gamma x) = \gamma g(x)$ for all $\gamma \in \Gamma$, $x \in \mathbb{R}^n$ from some neighbourhood of zero.
- Germs f and g are contact equivalent if there is a local diffeomorphism X and linear maps $S(x)\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$g(x) = S(x)f(X(x)).$$

- If f and g are Γ-equivariant we say that they are Γ-contact equivalent if, moreover, X is Γ-equivariant and $S(\gamma x)\gamma = \gamma S(x)$, for all $\gamma \in \Gamma$, $x \in \mathbb{R}^n$
- Equivalence preserve qualitative properties of the solution set of $g = 0$ in a neighbourhood of 0.

$\mathbb{Z}_q$-equivariant germs I

- $\mathbb{Z}_q$ acts on $\mathbb{R}^2$ as rotations of the plane by multiples of $2\pi/q$.
- $\mathbb{Z}_q$ is isomorphic to the group of q-th roots of unity and we can identify $\mathbb{R}^2 \cong \mathbb{C}$.
- We consider just $q \geq 5$, cases $q \in \{1, 2, 3, 4\}$ are the so called strong resonances - each case has to be solved individually.
- Any Z_q-equivariant germ $g\colon \mathbb{C} \to \mathbb{C}$ can be expressed in the following form:

$$g(z) = K(u, v)z + L(u, v)\bar{z}^{q-1}, \tag{4}$$

where $u = z\bar{z}$, $v = z^q + \bar{z}^q$ and K and L are suitable complex-valued germs.

$\mathbb{Z}_q$-equivariant germs II

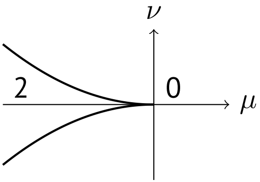
Theorem

Suppose that g has the form (4), where $K(0,0) = 0$. *If* $K_u L(0,0) \neq 0$, *then g is* $\mathbb{Z}_q$*-contact equivalent to*

$$h(z) = uz + \bar{z}^{q-1}$$

with universal unfolding

$$H(z, \mu, \nu) = (\mu + i\nu + u)z + \bar{z}^{q-1}. \tag{5}$$



Neimark-Sacker bifurcation

- Consider
$$\frac{\mathrm{d}u}{\mathrm{d}t} - F(u, \eta, \vartheta) = 0, \tag{6}$$
where $F\colon \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$ is a smooth function and η and ϑ are parameters.
- Suppose that (6) has a p-periodic solution u_0 for $\eta = \eta_0$ and $\vartheta = \vartheta_0$ and that its Floquet multipliers lying in the unit circle are 1, $\xi = \mathrm{e}^{i\frac{r}{q}2\pi}$, and $\overline{\xi} = \mathrm{e}^{-i\frac{r}{q}2\pi}$, where $r, q \in \mathbb{N}$ are coprime, and these eigenvalues are simple.
- Similarly to the case of two oscillators, Equation (6) defines an operator $\Phi\colon C^1_{qp} \times \mathbb{R}^3 \to C_{qp}$.

Lyapunov-Schmidt reduction and singularity theory

- Reduction of $\Phi = 0$ to $g(z, \bar{z}, \eta, \vartheta) = 0$, where $g\colon \mathbb{C} \times \mathbb{R}^2 \to \mathbb{C}$.
- The germ $g(z, \bar{z}, \eta_0, \vartheta_0)\colon \mathbb{C} \to \mathbb{C}$ is $\mathbb{Z}_q$-equivariant and $g(z, \bar{z}, \eta, \vartheta)$ is its (universal) unfolding.
- Aim: to find path $(\eta(\lambda), \vartheta(\lambda))$ such that it lies in the tongue for $\lambda > 0$ small enough.
- Solution: decide for which path $(\eta(\lambda), \vartheta(\lambda))$ the family of germs $g(z, \bar{z}, \eta(\lambda), \vartheta(\lambda))$ is point-wise $\mathbb{Z}_q$-contact equivalent to germs $H(z, \bar{z}, \Lambda(\lambda), 0)$, where Λ is a suitable reparameterization. $\rightsquigarrow$ recognition problem for one-parameter families of germs.
- Key result: There is a polynomial $p_0(\lambda)$ of order $q - 3$ whose coefficients are expressions containing derivatives of functions $\eta(\lambda), \vartheta(\lambda)$ with the following property:

Theorem

If $p_0 \equiv 0$, then $g(z, \bar{z}, \eta(\lambda), \vartheta(\lambda))$ is equivalent to $(u + \lambda)z + \bar{z}^{q-1}$.

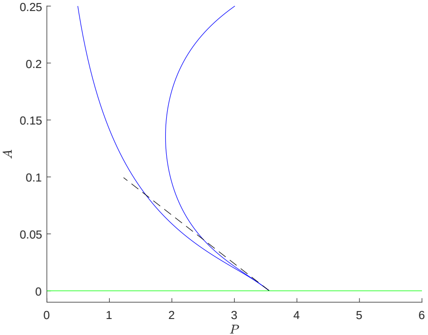
The path and the tangent vector

- If η and ϑ are parameters of universal unfolding, then the system $p_0 \equiv 0$ is solvable - it determines derivatives of functions $\eta(\lambda)$ and $\vartheta(\lambda)$ up to order $q-3$.
- A (polynomial) path $(\eta(\lambda), \vartheta(\lambda))$ satisfying these constraints has the property that it lies within the tongue for $\lambda > 0$ small enough.
- The first order condition (linear coefficient of p_0 is zero) gives

$$\frac{\partial^2}{\partial z \partial \lambda}\bigg|_{(0,\eta_0,\vartheta_0)} g(z, \eta(\lambda), \vartheta(\lambda)) = r \cdot \frac{\partial^3}{\partial z^2 \partial \bar{z}}\bigg|_{(0,\eta_0,\vartheta_0)} g(z, \eta(\lambda), \vartheta(\lambda)).$$

- The tangent vector to the tongue in the resonance point can be explicitly calculated from the last equation.

Josephson junctions



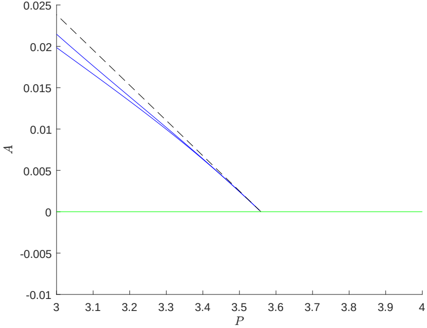


Figure: 2:7 resonance Arnold tongue with its tangent line. The boundary of the tongue is formed by the fold bifurcation curve of cycles and is shown in blue. The Neimar-Sacker bifurcation curve is shown in green. The tongue tangent at the point of resonance is represented by the black dashed line. On the right, the vicinity of the resonance point is zoomed in.

Notes on numerical methods

- The knowledge of tangent vector can simplify or even enable continuation of tongues
- We can start a continuation of limit cycle on the tangent line close to the Neimark-Sacker bifurcation curve via some continuation software.
- In the case of the Matcont software, the starting point is refined during the initialization of the continuation via Newton's method, and it is possible to achieve a true (r, q)-cycle inside the tongue in this way.
- This can be continued until the fold bifurcation of the limit cycle is found. Finally, from this point, we can compute the fold bifurcation curve of cycles, which will finally lead to the tongue boundary.

Remarks

- Clearly, this approach may involve some obstacles related to numerical computations, such as the choice of an appropriate step length on the tangent line or the problem of convergence from a point on the tangent line to the actual periodic solution inside the tongue, and it is important to verify the obtained results on other examples.
- All the results should be straightforwardly generalizable to delay differential equations, however the formulae will be considerably more complicated.

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